Quantum Transport and Boltzmann Operators[1]

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In this paper the transport of quantum particles in time-dependent random media is studied. In the white noise limit, a quantum model for collisions is obtained. At the level of Wigner equation, this limit is described by a linear Wigner-Boltzmann equation.

Nous étudions dans cet article le transport de particules quantiques dans un milieu aléatoire dépendant du temps. Dans la limite de bruit blanc, nous obtenons un modèle quantique de collison. Au niveau des équations de Wigner, la limite est donnée par une équation de Wigner-Boltzmann linéaire.

À Fredo. Frédéric Poupaud deceased October 13th, 2004.

1. INTRODUCTION

In this paper we investigate the asymptotic behavior of quantum particles dynamics in a random media. The random media is modeled by a random potential $V_t^\tau(x)$ which is time dependent. The small parameter τ represents the correlation time: $V_t^\tau(x)$, $V_s^\tau(y)$ are independent random variables as soon as $|t - s| \geq \tau$. Therefore the limit $\tau \to 0$ corresponds to a white noise limit. The amplitude of

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the potential is of order $1/\sqrt{\tau}$. We will prove that, starting from the classical Schrödinger picture to describe the evolution of quantum particles, the correlation time limit $\tau \to 0$ describes a Wigner-Boltzmann equation. This kind of problems belongs to the class of rigorous derivations of an irreversible dynamics from a reversible one. The effect of the scaling analyzed in this paper is that the random potential acts very strongly and, as a consequence, the process is very close to a Markovian dynamics with instantaneous generator. It could be possible to combine the whith noise limit introduced by the scaled random potential with the homogeneization given by the semiclassical limit, where the memory of the random potential is comparable with the full time scale. In our opinion, one of the advantages of doing the white noise limit is that we can recover the quantum effect in the Schrödinger formalism (we obtain a quantum Boltzmann-like equation in the Wigner formalism), while in combination with the semiclassical limit we find the classical description. Of course, the semiclassical analysis could be done in a second step. In the context of radiative transport theory (see [7,23], one of the main applications of this procedure is to study quantum effects in the description of the propagation of wave energy in scattering medium. Let us briefly summarize some contributions to the literature on this field.

H. Spohn derived in[25] the spatially homogeneous radiative transport equation starting from the Schrödinger equation for short times, for electrons moving through random impurities modeled by time-independent Gaussian potential. This result was generalized to higher-order correlation functions by T. Ho, L. Landau and A. Wilkins in.[18] In[14], the small time restriction was removed and the result was extended to more general initial data by L. Erdös and H. T. Yau, where the potential had no loss of memory. The above results can be considered in the framework of the weak coupling limit (see also[13]), but in[14] it is also analyzed the so-called low-density limit, which is the quantum analogue of the classical Lorentz gas. In the above results the proofs are mainly based on Neumann's series expansion for the solutions of the Schrödinger equation. In[3] time-dependent random potentials modeled by a Markov process in time are considered. Then, G. Bal, G. Papanicolaou and L. Ryzhik performed the radiative transport limit by constructing an approximate martingale for the random Wigner distribution, where the time scale of the memory of the random potential was comparable with the full time scale. F. Poupaud and A. Vasseur in[24] dealt with the same problem with rapidly decorrelating in time potential in combination with the semiclassical limit. The effective equation obtained in those papers is a classical linear Boltzmann equation. Our aim in this paper is different, while the techniques are close to those of.[24] We want to obtain a model for collisions at the quantum level starting from classical Schrödinger equations. We follow the general mathematical approach of.[24] In particular, the limit is performed directly on the equation and not on an explicit representation of the solution. The stochasticity in time of

the potential automatically implies the non self-correlation of particles paths. In [9,10], a non-commutative version of the entropy extremalization principle allows to construct new quantum hydrodynamic models founded in the moment method. The moment system is closed by a quantum (Wigner) distribution function which minimizes the entropy subject to the constraint that its moments are given and the resulting moment system involves nonlocal operators. In [11] P. Degond and C. Ringhofer generalize the previous results to the Boltzmann collision operator using nonlocal quantum entropy principles.The problem of finding diffusion models in quantum transport can be also considered in this context and plays an important role in a wide range of applications from which we mention the microelectronic devices. The motion of an ensemble of quantum particles interacting with a heat bath of oscillators in thermal equilibrium was modeled by A. O. Caldeira and A. J. Legget in [5], see also [12]. For open quantum systems, the analysis of dissipative transport equations with Fokker–Planck–type scattering mechanism was done in [1] by A. Arnold, J. L. López, P. A. Markowich and J. Soler in the Wigner function formalism [26] (level of the kinetic equation), see also [6]. At the level of the density operator the same problem has been recently studied by A. Arnold and C. Sparber in [2] and by J. L. López in [21] in the setting of (logarithmic) Schrödinger systems.

All the dissipative quantum models studied in the previous citations rely on *a priori* assumptions or principles. On the contrary, in this work, we start from Schrödinger equations and we rigorously derive a Wigner-Boltzmann equation, thus obtaining a dissipative quantum model.

The paper is structured as follows: In Section 2 we analyze the main features of the random potential leading to the white noise limit and to the quantum transport equations. Section 3 is devoted to deduce the Schrödinger and the Wigner equations as function of the correlation time parameter. In Section 4 we perform the white noise asymptotic limit and introduce the main results of this paper. We also discuss the simpler case in which the correlation of the random potential depends only on one variable. The conservation of the positiveness for the density matrix is analyzed. Finally, Section 5 concerns the proofs of the results obtained in the previous sections.

2. THE RANDOM POTENTIAL

At the quantum level, the dynamics of particles is governed by a potential $V^\tau(t, x)$ which is assumed to be a real function of time and space variables $(t, x) \in \mathbb{R}^{1+D}$, where D is the space dimension. It is for instance the sum of a potential due to an applied bias and of a random potential due to inhomogenity of the media (impurities or phonons in a semiconductor for instance). More precisely, we suppose that the potential has the following form

$$V^\tau(t, x) = V(t, x) + U^\tau(t, x), \text{ with}$$

$$U^{\tau}(t, x) = \frac{1}{\sqrt{\tau}} U\left(\frac{t}{\tau}, x\right), \quad \forall (t, x) \in \mathbb{R}^{1+D}, \tag{1}$$

where V is a measurable deterministic function (actually the expectation of V^{τ}) and U is a measurable random function. The assumptions on these functions are listed below.

- There is some positive constant C_{∞} such that

$$|V(t, x)| \leq C_{\infty}, \quad |U(t, x)| \leq C_{\infty} \text{ almost surely}, \forall (t, x) \in \mathbb{R}^{1+D}. \tag{2}$$

- The expectation of the random potential vanishes,

$$\mathbb{E}(U(t, x)) = 0, \quad \forall (t, x) \in \mathbb{R}^{1+D}. \tag{3}$$

- We have the following Markov property

$$U(t, x), \ U(s, y) \text{ are independent random variables} \tag{4}$$

for all $t, s \in \mathbb{R}$ such that $|t - s| \geq 1$.
- We also impose that the random function U is stationary with respect to time. It means that there is a measurable bounded function $R = R(t, x, y)$ such that

$$\mathbb{E}(U(t, x)U(s, y)) = R(t - s, x, y), \forall (t, x) \in \mathbb{R}^{1+D}, \ \forall (s, y) \in \mathbb{R}^{1+D}. \tag{5}$$

We remark that due to the Markov property (4) and to (3), the support of R lies in $[-1, 1] \times \mathbb{R}^{2D}$. We also have that R verifies : $R(t, x, y) = R(-t, y, x)$. It allows to define the symmetric, bounded, real function

$$S(x, y) = \int_{-1}^{1} R(t, x, y) \, dt = \int_{\mathbb{R}} R(t, x, y) \, dt. \tag{6}$$

We need some regularity assumptions on R. Indeed, we shall assume that $\hat{R}$ is a measure satisfying

$$\int_{\mathbb{R}} \int_{\mathbb{R}^{2D}} |\hat{R}(t, p, q)| \, |q| \, (1 + |q| + |p|) \, dp \, dq \, dt \leq C \tag{7}$$

for some constant $C > 0$, where

$$\hat{R}(t, p, q) = \int_{\mathbb{R}^D \times \mathbb{R}^D} R(t, x, y) \, e^{-i(p \cdot x + q \cdot y)} \, dp \, dq.$$

A particular case is when the random potential is also stationary with respect to space. In this case we have for a measurable bounded function $Q = Q(t, x)$

$$\mathbb{E}(U(t, x)U(s, y)) = Q(t - s, x - y), \quad \forall (t, x) \in \mathbb{R}^{1+D}, \ \forall (s, y) \in \mathbb{R}^{1+D}. \tag{8}$$

Then $\hat{R}(t, p, q) = (2\pi)^D \hat{Q}(t, p) \delta(q + p)$ and assumption (7) is reduced to

$$\int_{\mathbb{R}} \int_{\mathbb{R}^D} |\hat{Q}(t, p)| \, |p| \, (1 + |p|) \, dp \, dt \leq C. \tag{9}$$

3. SCHRÖDINGER AND WIGNER EQUATIONS

This Section is concerned with the asymptotic behavior when $\tau \to 0$ of the solutions of the following (normalized) Schrödinger equations

$$i \, \frac{\partial}{\partial t} \psi_n^\tau = -\frac{1}{2} \Delta_x \psi_n^\tau + V^\tau(t, x) \psi_n^\tau, \quad t \in \mathbb{R}, \ x \in \mathbb{R}^D, \ n = 1, 2, \dots \tag{10}$$

$$\psi_n^\tau(0, x) = \psi_n^I(x), \quad x \in \mathbb{R}^D, \ n = 1, 2, \dots \tag{11}$$

Here, the random potential V^τ satisfies the hypotheses of the previous Section and τ is a small parameter which represents the correlation time of V^τ.

We use the mixed state approach. The initial data are assumed to form an orthonormal system of $L^2(\mathbb{R}^D)$. It classically results that for all time $t \in \mathbb{R}$, the system $(\psi_n^\tau(t))_{n=1,2,\dots}$ is also orthonormal

$$\int_{\mathbb{R}^D} \psi_n^\tau(t, x) \, \overline{\psi_m^\tau(t, x)} \, dx = \delta_{nm}, \tag{12}$$

for $t \in \mathbb{R}$, and $n, m = 1, 2, \dots$ where δ_{nm} stands for the Kronecker delta symbol. To each index n there corresponds an occupation probability λ_n, while the state of the particle is described by the wave function ψ_n^τ. We assume that

$$\lambda_n \geq 0, \quad \sum_{n=1}^{\infty} \lambda_n = 1. \tag{13}$$

As in [20], we introduce the time-dependent Wigner function associated with the mixed state

$$w^\tau(t, x, \xi) = \sum_{n=1}^{\infty} \lambda_n \frac{1}{(2\pi)^D} \int_{\mathbb{R}^D} \psi_n^\tau \left(t, x + \frac{y}{2}\right) \overline{\psi_n^\tau \left(t, x - \frac{y}{2}\right)} e^{-iy.\xi} \, dy \tag{14}$$

and the initial Wigner function

$$w^I(x, \xi) = \sum_{n=1}^{\infty} \lambda_n \frac{1}{(2\pi)^D} \int_{\mathbb{R}^D} \psi_n^I \left(x + \frac{y}{2}\right) \overline{\psi_n^I \left(x - \frac{y}{2}\right)} e^{-iy.\xi} \, dy. \tag{15}$$

We refer to [15,16,22,20] for properties of Wigner functions. We only emphasize that the weak limit of w^τ allows to determine the limit of observables of quantum mechanics. We have (see [16], for instance)

Proposition 3.1. *Assume that the functions ψ_n^τ solve (10), (11) with initial data that form an orthonormal system. Then, the Wigner functions w^τ ($\tau > 0$) defined by (14) with occupation probabilities satisfying (13) are real functions. Also, they lie in a bounded set of $L^\infty(\mathbb{R}; L^2(\mathbb{R}^{2D}))$*

$$\|w^\tau(t)\|_{L^2(\mathbb{R}^{2D})} \leq \sqrt{C_0}, \quad \forall \tau > 0, \ \forall t \in \mathbb{R}, \quad \textit{almost surely,} \tag{16}$$

with $C_0 = \sum_{n=1}^{\infty} \lambda_n{}^2$. The probability density defined by

$$n^\tau(t, x) = \sum_{n=1}^{\infty} \lambda_n |\psi_n^\tau(t, x)|^2 = \int_{\mathbb{R}^D} w^\tau(t, x, \xi) \, d\xi \tag{17}$$

is bounded in $L^\infty(\mathbb{R}; L^1(\mathbb{R}^D))$.

In order to derive the evolution equation satisfied by the Wigner function, we need to introduce the following pseudo-differential operators. For a given bounded measurable function Φ, we define the operator

$$\theta[\Phi] = i\left(\Phi\left(x + \frac{D_\xi}{2}\right) - \Phi\left(x - \frac{D_\xi}{2}\right)\right). \tag{18}$$

This operator is explicitly given by

$$\theta[\Phi](\eta) := \frac{i}{(2\pi)^D} \int_{\mathbb{R}^D} \left(\Phi\left(t, x + \frac{y}{2}\right) - \Phi\left(t, x - \frac{y}{2}\right)\right) \mathcal{F}_{v \to y}(\eta(x, v)) \, e^{iy.\xi} \, dy,$$

where $\mathcal{F}_{v \to y}$ is the Fourier transform between the dual variables v and y

$$\mathcal{F}_{v \to y}(\eta(v)) := \int_{\mathbb{R}^D} \eta(v) \, e^{-iv.y} \, dv.$$

This operator is bounded on $L^2(\mathbb{R}^D)$. Its norm in the space $\mathcal{L}(L^2(\mathbb{R}^D))$ of linear operators on L^2, denoted by $|||\theta[\Phi]|||$, is bounded by $2\|\Phi\|_{L^\infty(\mathbb{R}^D)}$.

We introduce

$$\theta_t^\tau := \theta[V^\tau(t)] = i\left(V^\tau\left(t, x + \frac{D_\xi}{2}\right) - V^\tau\left(t, x - \frac{D_\xi}{2}\right)\right). \tag{19}$$

Thanks to (2), its norm is bounded by

$$|||\theta_t^\tau||| \leq 2\,\|V^\tau\|_{L^\infty(\mathbb{R}^{D+1})} \leq 2\,C_\infty\left(1 + \frac{1}{\sqrt{\tau}}\right). \tag{20}$$

If $\mathcal{V}_t^\tau$ denotes the Fourier transform of $U^\tau(t)$ with respect to the space variable (it is a tempered distribution) we obtain

$$\theta[U^\tau(t)](\eta) = \frac{i}{(2\pi)^D} \int_{\mathbb{R}^D} \mathcal{V}_t^\tau(p) \left(\eta\left(x, \xi + \frac{p}{2}\right) - \eta\left(x, \xi - \frac{p}{2}\right)\right) e^{ix.p} \, dp. \tag{21}$$

In the above formula the integral has to be understood as a duality between a distribution and a function. The expectation $\mathbb{E}(\mathcal{V}_t^\tau(p)\,\mathcal{V}_s^\tau(q))$, which will be useful in the next section, can be obtained from the following computation

$$\mathbb{E}(\mathcal{V}_t^\tau(p)\,\mathcal{V}_s^\tau(q)) = \mathcal{F}_{x\to p}\mathcal{F}_{y\to q}\mathbb{E}(U^\tau(t,x)U^\tau(s,y))$$

$$= \frac{1}{\tau}\mathcal{F}_{x\to p}\mathcal{F}_{y\to q} R\left(\frac{t-s}{\tau}, x, y\right) = \frac{1}{\tau}(2\pi)^D \hat{R}\left(\frac{t-s}{\tau}, p, q\right). \quad (22)$$

This result can be deduced by using (5) and defining $\hat{R}(t,p,q) = \mathcal{F}_{x\to p}\mathcal{F}_{y\to q} R(t,x,y)$. Note that $\hat{R}$ verifies $\hat{R}(t,p,q) = \hat{R}(-t,q,p)$. We now introduce the Wigner equation. We have (see [22,20,16])

Proposition 3.2. *Under the same hypotheses as in Proposition 3.1, the Wigner functions w^τ solve the following Wigner equation*

$$\frac{\partial}{\partial t}w^\tau + \xi.\nabla_x w^\tau = \theta_t^\tau(w^\tau), \quad t \in \mathbb{R}, \ x \in \mathbb{R}^D, \ \xi \in \mathbb{R}^D, \quad (23)$$

$$w^\tau(0,x,\xi) = w^I(x,\xi), \qquad x \in \mathbb{R}^D, \ \xi \in \mathbb{R}^D, \quad (24)$$

where the operator θ_t^τ is defined by (19). For all time $t \in \mathbb{R}$, θ_t^τ is a bounded skew operator on $L^2(\mathbb{R}^{2D})$ which satisfies (20).

4. WHITE NOISE LIMIT

The aim of this Section is to determine the asymptotic behavior of the expectation value $\mathbb{E}(w^\tau)$ when $\tau \to 0$. Our main result is the following

Theorem 4.1. *Assume that the random potential satisfies the assumptions of Section 2. Also assume that the functions ψ_n^τ solve (10), (11) with initial data which form an orthonormal system. Suppose that the occupation probabilities satisfy (13) and that the initial data ψ_n^I are deterministic.*
 Then, when the parameter $\tau \to 0$ we have

$$\mathbb{E}(w^\tau) \to w^0 \quad in \quad C^0([0,T]; L^2(\mathbb{R}^{2D}) - weak), \qquad for \ any \ T > 0.$$

We also have $w^0(t=0) = w^I$ where w^I is defined by (15). Moreover, w^0 is the solution of the following Wigner–Boltzmann equation

$$\frac{\partial}{\partial t}w^0(t,x,\xi) + \xi.\nabla_x w^0(t,x,\xi) = \theta[V(t)](w^0)(t,x,\xi)$$

$$- \int_{\mathbb{R}^D} K_1(x,\xi'-\xi)w^0(t,x,\xi')\,d\xi' + \int_{\mathbb{R}^D} K_2(x,\xi'-\xi)\,w^0(t,x,\xi')\,d\xi' \quad (25)$$

for $t > 0$ and $(x, \xi) \in \mathbb{R}^{2D}$, where

$$K_1(x, p) = \int_{\mathbb{R}^D} S\left(x + \frac{x'}{2}, x + \frac{x'}{2}\right) \cos(p \cdot x') \, dx', \qquad (26)$$

$$K_2(x, p) = \int_{\mathbb{R}^D} S\left(x - \frac{x'}{2}, x + \frac{x'}{2}\right) \cos(p \cdot x') \, dx' \qquad (27)$$

and S is defined by (6). In the particular case where the potential is stationary with respect to space, see (8), equation (25) reads

$$\frac{\partial}{\partial t} w^0(t, x, \xi) + \xi . \nabla_x w^0(t, x, \xi) = \theta[V(t)](w^0)(t, x, \xi) - \Lambda \, w^0(t, x, \xi)$$

$$+ \int_{\mathbb{R}^D} k(\xi' - \xi) w^0(t, x, \xi') \, d\xi' \qquad (28)$$

for $t > 0$ and $(x, \xi) \in \mathbb{R}^{2D}$, where

$$k(p) = \int_{\mathbb{R}^D} \int_{\mathbb{R}} Q(\sigma, x) e^{-ip \cdot x} \, d\sigma \, dx \geq 0,$$

$$\Lambda = (2\pi)^D \int_{\mathbb{R}^D} k(p) \, dp = \int_{\mathbb{R}} Q(\sigma, 0) \, d\sigma,$$

where Q is defined by (8).

The above equation makes the operator

$$\eta \mapsto L(\eta) : \xi \to \int_{\mathbb{R}^D} k(\xi' - \xi) \eta(\xi') \, d\xi' - \Lambda \eta(\xi)$$

$$= \int_{\mathbb{R}^D} k(\xi' - \xi)(\eta(\xi') - \eta(\xi)) \, d\xi'$$

to appear. This operator can be seen as a linear Boltzmann operator because k is nonnegative. In particular it is dissipative for the L^2-norm because

$$\forall \eta \in L^2(\mathbb{R}^D), \quad \int_{\mathbb{R}^D} L(\eta)(\xi) \, \eta(\xi) \, d\xi$$

$$= -\frac{1}{2} \int_{\mathbb{R}^{2D}} k(\xi' - \xi) (\eta(\xi') - \eta(\xi))^2 \, d\xi' \, d\xi \leq 0.$$

Indeed, with the notation of [24], we have $k(p) = 2\pi \mathcal{R}(0, p)$, so it results from Lemma 4.5 of [24] that k is even, nonnegative and continuous. Thus, (28) can be seen as a quantum Boltzmann equation.

This interpretation is less obvious in the case of equation (25). It appears more clearly in the equivalent formulation in terms of the density matrix. The

density matrix $\mathcal{D}^\tau$ is a self–adjoint, nonnegative, time dependent operator which acts on $L^2(\mathbb{R}^D)$. The integral kernel of this operator reads

$$\rho^\tau(t, x, y) = \sum_{n=1}^\infty \lambda_n \psi_n^\tau(t, x) \, \overline{\psi_n^\tau(t, y)}. \tag{29}$$

The kernel ρ^τ is related to the Wigner function by the identities

$$w^\tau(t, x, \xi) = \frac{1}{(2\pi)^D} \int_{\mathbb{R}^D} \rho^\tau(t, x - y/2, x + y/2) e^{iy \cdot \xi} \, dy$$

and

$$\rho^\tau(t, x, y) = \int_{\mathbb{R}^D} w^\tau \left(t, \frac{x+y}{2}, \xi \right) e^{-i(y-x) \cdot \xi} \, d\xi. \tag{30}$$

Remark that we have $\|\rho^\tau\| = \sqrt{(2\pi)^D} \|w^\tau\|$. Then, Theorem 4.1 implies that

$$\mathbb{E}(\rho^\tau) \to \rho^0 \qquad \text{in } C^0([0, T]; L^2(\mathbb{R}^{2D}) - weak), \qquad \text{for any } T > 0.$$

We now use the relation between the Wigner transform and the density matrix in the Wigner equation to deduce that ρ^0 is the solution of

$$i \frac{\partial}{\partial t} \rho^0(t, x, y) = \left[-\frac{1}{2} \Delta + V(t), \rho^0(t) \right] (x, y)$$

$$- \frac{i}{2} (S(x, x) + S(y, y) - 2S(x, y)) \, \rho^0(t, x, y), \tag{31}$$

where $[A, B] := AB - BA$ denotes the commutator of the operators A, B and S is defined in (6). The main difficulty in deriving (31) comes from the terms involving the kernels K_1 and K_2. More precisely, a short computation shows that

$$\int_{\mathbb{R}^D \times \mathbb{R}^D} K_1 \left(\frac{x+y}{2}, \xi' - \xi \right) w \left(\frac{x+y}{2}, \xi' \right) e^{-i(y-x) \cdot \xi} \, d\xi \, d\xi'$$

$$= \frac{1}{2} (S(x, x) + S(y, y)) \rho(x, y)$$

and

$$\int_{\mathbb{R}^D \times \mathbb{R}^D} K_2 \left(\frac{x+y}{2}, \xi' - \xi \right) w \left(\frac{x+y}{2}, \xi' \right) e^{-i(y-x) \cdot \xi} \, d\xi \, d\xi'$$

$$= S(x, y) \rho(x, y).$$

In view of assumptions (2), (3), (4) and (5), the function $R = R(t, x, y)$ is bounded and has a support embedded in $\{t \in [-1, 1]\}$. Therefore $t \mapsto R(t, x, y)$ is integrable for a.e. $(x, y) \in \mathbb{R}^{2D}$. Then using Lemma 3.1 of [24] we have

$$S(x, y) = \int_{\mathbb{R}} R(t, x, y) \, dt = \lim_{L \to \infty} \int_{-L}^{L} \int_{-L}^{L} R(t - \sigma, x, y) \, dt \, d\sigma \qquad a.e.$$

$$= \lim_{L \to \infty} \mathbb{E}(\mathcal{U}_L(x)\mathcal{U}_L(y)) \quad a.e. \tag{32}$$

with $\mathcal{U}_L(x) = \dfrac{1}{\sqrt{L}} \displaystyle\int_{-L}^{L} U(t, x)$. It follows that

$$S(x, x) = \lim_{L \to \infty} \mathbb{E}(\mathcal{U}_L(x)^2) \geq 0 \quad a.e.$$

$$S(x, x) + S(y, y) - 2S(x, y) = \lim_{L \to \infty} \mathbb{E}((\mathcal{U}_L(x) - \mathcal{U}_L(y))^2) \geq 0 \quad a.e. \tag{33}$$

We will see that the last inequality can be interpreted as a dissipativeness property of equation (31). The following result is also useful.

Lemma 4.1.   *For any symmetric, trace class operator $\mathcal{D}$ on $L^2(\mathbb{R}^D)$ whose kernel is $\rho = \rho(x, y)$, let $\mathcal{S}(\mathcal{D})$ be the operator whose kernel is $(2\pi)^D S(x, y) \rho(x, y)$. Then, $\mathcal{S}$ is linear and continuous on the space of symmetric, trace class operators and preserves nonnegativeness.*

The function $S = S(x, y)$ is bounded, real and symmetric with respect to (x, y), then the only point to check is the nonnegativeness of $\mathcal{S}(\mathcal{D})$ when $\mathcal{D} \geq 0$. We have for any $\varphi \in L^2(\mathbb{R}^D)$

$$< \varphi, \mathcal{S}(\mathcal{D})\varphi > = (2\pi)^D \int_{\mathbb{R}^{2D}} \overline{\varphi(x)} S(x, y) \rho(x, y) \varphi(y) \, dx \, dy$$

$$= (2\pi)^D \lim_{L \to \infty} \mathbb{E}\left(\int_{\mathbb{R}^{2D}} \overline{\varphi(x)} \mathcal{U}_L(x) \, \rho(x, y) \mathcal{U}_L(y) \varphi(y) \, dx \, dy \right)$$

$$= (2\pi)^D \lim_{L \to \infty} \mathbb{E}(< \mathcal{U}_L \varphi, \mathcal{D}\mathcal{U}_L \varphi >) \geq 0$$

where the second equality has been obtained by using the Dominated Convergence Theorem. This ends the proof of the Lemma.

Let $\mathcal{T}$ be the multiplication operator of $\mathcal{L}(L^2(\mathbb{R}^D))$ given by

$$\mathcal{T}\varphi(x) = (2\pi)^D S(x, x)\varphi(x), \quad for \ a.e. \ x \in \mathbb{R}^D, \quad \forall \varphi \in L^2(\mathbb{R}^D).$$

Let $H(t)$ be the self–adjoint operator $H(t) = -\frac{1}{2}\Delta \cdot + V(t) \cdot$ and $\mathcal{D}^0$ be the density matrix corresponding to the kernel ρ^0. It results from (31) that

Proposition 4.3.   *Under the same assumptions as in Theorem 4.1, the density matrix $\mathcal{D}^0$ corresponding to the limit kernel ρ^0 solves the Von Neuman-Boltzmann equation*

$$\frac{d}{dt}\mathcal{D}^0(t) = -i[H(t), \mathcal{D}^0(t)] + \mathcal{S}(\mathcal{D}^0(t)) - \frac{1}{2}(\mathcal{T}\mathcal{D}^0(t) + \mathcal{D}^0(t)\mathcal{T}) \tag{34}$$

for all $t > 0$. This equation generates a continuous nonautonomous group on the space of symmetric trace class operators. It preserves the trace and

nonnegativeness. Moreover it is dissipative for the norm $\|\mathcal{D}\|_2 = \sqrt{tr(\mathcal{D}\mathcal{D}^*)}$ in the sense that $\|\mathcal{D}^0(t)\|_2$ is a nonincreasing function of time.

The first assertion is obvious because the operator $\mathcal{D} \mapsto \mathcal{S}(\mathcal{D}) - \frac{1}{2}(\mathcal{T}\mathcal{D} + \mathcal{D}\mathcal{T})$ is a bounded linear operator on the space of symmetric trace class operators and we have classically that the Von Neuman operator $\mathcal{D} \mapsto -i[H(t), \mathcal{D}]$ generates a continuous non autonomous group. Concerning the trace conservation we have

$$\frac{d}{dt}\text{tr}(\mathcal{D}(t)) = \text{tr}\left(\mathcal{S}(\mathcal{D}(t)) - \frac{1}{2}(\mathcal{T}\mathcal{D}(t) + \mathcal{D}(t)\mathcal{T})\right)$$

and we easily check using the kernels of the operators that for any $\mathcal{D}$ $\text{tr}(\mathcal{S}(\mathcal{D}) - \frac{1}{2}(\mathcal{T}\mathcal{D} + \mathcal{D}\mathcal{T})) = 0$.

It remains to prove that $\mathcal{D}(t)$ remains nonnegative if $\mathcal{D}(0) \geq 0$. Let us introduce the group $\mathcal{G}_{t,s}$ generated by

$$\mathcal{D} \mapsto -i[H(t), \mathcal{D}] - \frac{1}{2}(\mathcal{T}\mathcal{D} + \mathcal{D}\mathcal{T}).$$

We first prove that this group preserves nonnegativeness. Let $\mathcal{D}_I$ be a trace class, symmetric nonnegative operator. Then there is an orthonormal system $(\varphi_n)_{n \in \mathbb{N}}$ of $L^2(\mathbb{R}^D)$ and a real sequence $(\lambda_n)_{n \in \mathbb{N}}$ satisfying $\lambda_n \geq 0$, $\sum_{n \in \mathbb{N}} \lambda_n < \infty$ such that $\mathcal{D}_I = \sum_{n \in \mathbb{N}} \lambda_n \varphi_n \otimes \varphi_n$. Let $(\psi_n(t))_{n \in \mathbb{N}}$ be the solutions of the Schrödinger equation

$$i\frac{d}{dt}\psi_n(t) = H(t)\psi_n(t) + i\mathcal{T}\psi_n(t), \qquad \psi_n(s) = \varphi_n.$$

Then, it is easy to check that $\mathcal{G}_{t,s}\mathcal{D}_I = \sum_{n \in \mathbb{N}} \lambda_n \psi_n(t) \otimes \psi_n(t)$, therefore $\mathcal{G}_{t,s}\mathcal{D}_I$ is also nonnegative. Since the operator $\mathcal{D} \mapsto \mathcal{S}(\mathcal{D})$ is Lipschitz continuous, the solution $\mathcal{D}(t)$ of equation (34) is given by the limit of the sequence

$$\mathcal{D}_1(t) = \mathcal{G}_{t,0}\mathcal{D}(0),$$

$$\mathcal{D}_{n+1}(t) = \mathcal{G}_{t,0}\mathcal{D}(0) + \int_0^t \mathcal{G}_{t,s}\mathcal{S}(\mathcal{D}_n(s))\,ds.$$

Using Lemma 4.1 and the fact that $\mathcal{G}_{t,s}$ preserves nonnegativeness, we obtain that $\mathcal{D}_n(t)$ is a sequence of nonnegative operators if $\mathcal{D}(0)$ is a trace class symmetric nonnegative operator. It results that the limit $\mathcal{D}(t)$ is also nonnegative.

The last point concerns dissipativeness. We compute

$$\frac{d}{dt}\mathcal{D}(t)^2 = -i([H(t), \mathcal{D}(t)]\mathcal{D}(t) + \mathcal{D}(t)[H(t), \mathcal{D}(t)]) + \mathcal{E}(t) = \mathcal{E}(t)$$

with $\mathcal{E}(t) = \mathcal{D}(t)\left(\mathcal{S}(\mathcal{D}(t)) - \frac{1}{2}(\mathcal{T}\mathcal{D}(t) + \mathcal{D}(t)\mathcal{T})\right)$

$$+ \left(\mathcal{S}(\mathcal{D}(t)) - \frac{1}{2}(\mathcal{T}\mathcal{D}(t) + \mathcal{D}(t)\mathcal{T})\right)\mathcal{D}(t).$$

The kernel of the operator $\mathcal{E}(t)$ is given by

$$-\frac{(2\pi)^D}{2} \int_{\mathbb{R}^D} \rho(t,x,z)(S(z,z)+S(y,y)-2S(z,y))\rho(t,z,y)$$

$$+ \rho(t,x,z)(S(z,z)+S(x,x)-2S(x,y))\rho(t,z,y)\,dz.$$

Therefore

$$\frac{d}{dt}\|\mathcal{D}(t)\|_2{}^2 = \frac{d}{dt}\mathrm{tr}(\mathcal{D}(t)^2) = \mathrm{tr}(\mathcal{E}(t))$$

$$= -\frac{(2\pi)^D}{2} \int_{\mathbb{R}^{2D}} \rho(t,x,z)(S(z,z)+S(x,x)-2S(z,x))\rho(t,z,x)\,dz\,dx$$

$$= -\frac{(2\pi)^D}{2} \int_{\mathbb{R}^{2D}} (S(z,z)+S(x,x)-2S(z,x))\,|\rho(t,z,x)|^2\,dz\,dx \le 0,$$

being the nonpositiveness due to (4.1). This ends the proof.

Remark 4.1. *The first idea for proving Proposition 4.3 is trying to put (34) in the Lindblad form, cf* [1,19]. *It consists of finding operators $(U_n)_{n\ge 1}$ such that*

$$S(\mathcal{D}) = \sum_{n=1}^{\infty} U_n \mathcal{D} U_n^*, \qquad \mathcal{T} = \sum_{n=1}^{\infty} U_n^* U_n. \tag{35}$$

The evolution equations in Lindblad form are dissipative in the space of trace–class operators and their quantum entropy grows, which is related to the irreversibility properties of the evolution equation. Lindblad's form also implies the conservation of positiveness, even gives rise to complete positiveness of the evolution semigroup.

This seem to be more complicated in our context than tryng to give a direct proof of Proposition 4.1. However, another possibility is to obtain S and $\mathcal{T}$ as a limit of operators of Lindblad form where the sums are replaced by expectations. Actually, using (32) we have

$$S(\mathcal{D}) = \lim_{L\to\infty} \mathbb{E}(U_L \mathcal{D} U_L), \qquad \mathcal{T} = \lim_{L\to\infty} \mathbb{E}(U_L{}^2),$$

where U_L is the self adjoint random operator corresponding to the multiplication by the real function $\mathcal{U}_L$. But the proofs in this approach are not simpler.

5. PROOFS

The rest of this paper is devoted to the proof of Theorem 4.1. From now on $\|.\|$ denotes the norm of $L^2(\mathbb{R}^{2D})$ and τ is assumed to satisfy $0 < \tau \le 1$. We also use the notation $O(\beta)$ for L^2-functions which are bounded in $L^2(\mathbb{R}^{2D})$ by $C\,\beta$ where C is a positive constant which is uniform with respect to the time t, the parameter τ and the random variable.

One of the main ingredients in the determination of the asymptotic behavior of w^τ is the Duhamel formula. We first introduce the unitary group on $L^2(\mathbb{R}^D)$, $(S_t)_{t\in\mathbb{R}}$, generated by the infinitesimal generator $\xi.\nabla_x$:

$$\forall \eta \in L^2(\mathbb{R}^D), \qquad S_t(\eta)(x,\xi) = \eta(x - t\,\xi, \xi), \quad x \in \mathbb{R}^D, \ \xi \in \mathbb{R}^D. \quad (36)$$

If w^τ is a solution of (23), (24) we obtain

$$w^\tau(t) = S_s w^\tau(t - s) + \int_0^s S_\sigma \theta^\tau_{t-\sigma}(w^\tau(t - \sigma)) \, d\sigma. \quad (37)$$

In particular, w^τ can be obtained as the fixed point of the map

$$w \mapsto S_t w^I + \int_0^t S_\sigma \theta^\tau_{t-\sigma}(w(t - \sigma)) \, d\sigma.$$

If the initial data is assumed to be independent upon the random potential, this formula shows that $w^\tau(t)$ depends only on V_s^τ for $s \in [0, t]$ if $t \geq 0$ (or for $s \in [t, 0]$ if $t \leq 0$). In view of the assumption (4), it follows

Lemma 5.2. *Assume that w_I is a deterministic function. Then, for all x, ξ, $y \in \mathbb{R}^D$, $t \geq 0$ and $s \geq t + \tau$, the functions $w^\tau(t, x, \xi)$ and $V_s^\tau(y)$ are independent random variables.*

We also have

Lemma 5.3. *Assume that $V_t^\tau(y)$ and $\eta(x,\xi)$ are independent random variables for all y, x, $\xi \in \mathbb{R}^D$. Then $\mathbb{E}(\theta_t^\tau(\eta)) = \theta[V](\mathbb{E}(\eta))$.*

Last lemma is a direct consequence of the definition (18) and also of (1) and (3). Also, a combination of the Duhamel formula (37) together with (20) and (16) allows to obtain the following useful estimate

$$\|w^\tau(t) - S_s w^\tau(t - s)\| \leq 2 C_0 C_\infty \left(1 + \frac{1}{\sqrt{\tau}}\right) s,$$

which implies

$$S_s w^\tau(t - s) = w^\tau(t) + O\left(\frac{s}{\sqrt{\tau}}\right). \quad (38)$$

We are now ready to use the strategy of $^{(24)}$ based on the use of the Duhamel formula and the time mixing properties. Taking the expectation of (23) we get

$$\frac{\partial}{\partial t} \mathbb{E}(w^\tau) + \xi.\nabla_x \mathbb{E}(w^\tau) = \mathbb{E}(\theta_t^\tau(w^\tau)), \quad t \in \mathbb{R}, \ x \in \mathbb{R}^D, \ \xi \in \mathbb{R}^D.$$

Thanks to Lemmas 5.2 and 5.3, we have for $t \geq \tau$

$$\mathbb{E}(\theta_t^\tau w^\tau(t)) = \mathbb{E}(\theta_t^\tau S_\tau w^\tau(t-\tau)) + \int_0^\tau \mathbb{E}(\theta_t^\tau S_\sigma \theta_{t-\sigma}^\tau w^\tau(t-\sigma)) \, d\sigma$$

$$= \theta[V(t)](\mathbb{E}(S_\tau w^\tau(t-\tau))) + \int_0^1 \mathbb{E}(\tau \, \theta_t^\tau S_{\sigma\tau} \theta_{t-\sigma\tau}^\tau w^\tau(t-\sigma\tau)) \, d\sigma.$$

We first remark that due to (38) we have

$$S_\tau w^\tau(t-\tau) = w^\tau(t) + O(\sqrt{\tau}), \quad w^\tau(t-\sigma\tau) = S_{(2-\sigma)\tau} w(t-2\tau) + O(\sqrt{\tau}).$$

The operators $\theta[V(t)]$ and $\tau \, \theta_t^\tau S_{\sigma\tau} \theta_{t-\sigma\tau}^\tau$ are of order 1 in $\mathcal{L}(L^2(\mathbb{R}^D))$. Therefore, we get for $t \geq 2\tau$

$$\mathbb{E}(\theta_t^\tau w^\tau(t)) = \theta[V(t)](\mathbb{E}(w^\tau(t)))$$

$$+ \int_0^1 \mathbb{E}(\tau \, \theta[U^\tau(t)] S_{\sigma\tau} \theta[U^\tau(t-\sigma\tau)] S_{(2-\sigma)\tau} w^\tau(t-2\tau)) \, d\sigma + O(\sqrt{\tau}).$$

For the second term we use again Lemma 5.2 to obtain

$$\mathbb{E}(\theta_t^\tau w^\tau(t)) = \theta[V(t)](\mathbb{E}(w^\tau(t)))$$

$$+ \int_0^1 \mathbb{E}(\tau \, \theta[U^\tau(t)] S_{\sigma\tau} \theta[U^\tau(t-\sigma\tau)] S_{-\sigma\tau}) \mathbb{E}(S_{2\tau} w^\tau(t-2\tau)) \, d\sigma + O(\sqrt{\tau})$$

$$= \theta[V(t)](\mathbb{E}(w^\tau(t)))$$

$$+ \int_0^1 \mathbb{E}(\tau \, \theta[U^\tau(t)] S_{\sigma\tau} \theta[U^\tau(t-\sigma\tau)] S_{-\sigma\tau}) \, d\sigma \, \mathbb{E}(w^\tau(t)) + O(\sqrt{\tau}).$$

We summarize these results in the following

Lemma 5.4. *Let w^τ be the Wigner functions defined in Propositions 3.1 and 3.2. Then, we have for $t \geq 2\tau$*

$$\frac{\partial}{\partial t} \mathbb{E}(w^\tau(t)) + \xi.\nabla_x \mathbb{E}(w^\tau(t)) = \theta[V(t)](\mathbb{E}(w^\tau(t)))$$

$$+ L_t^\tau(\mathbb{E}(w^\tau(t))) + O(\sqrt{\tau}), \qquad (39)$$

where the deterministic operator L_t^τ, defined by

$$L_t^\tau = \int_0^1 \mathbb{E}(\tau \, \theta[U^\tau(t)] S_{\sigma\tau} \theta[U^\tau(t-\sigma\tau)] S_{-\sigma\tau}) \, d\sigma \qquad (40)$$

on $L^2(\mathbb{R}^{2D})$ is uniformly bounded.

The first consequence of (39) is that the time derivative of $\mathbb{E}(w^\tau(t))$ is uniformly bounded with respect to τ and $t \in [2\tau, \infty)$, in the distributional sense. Let η be a test function and let us check the equicontinuity of $\int_{\mathbb{R}^{2D}} \mathbb{E}(w^\tau(t+2\tau))\eta \, dx \, d\xi$

on $t \in [0, \infty)$. We have

$$\int_{\mathbb{R}^{2D}} \mathbb{E}(w^\tau(t + 2\tau))\eta \, dx \, d\xi = \int_{\mathbb{R}^{2D}} \mathbb{E}(S_{2\tau} w^\tau(t))\eta \, dx \, d\xi + O(\sqrt{\tau})$$

$$= \int_{\mathbb{R}^{2D}} \mathbb{E}(w^\tau(t))S_{-2\tau}\eta \, dx \, d\xi + O(\sqrt{\tau})$$

$$= \int_{\mathbb{R}^{2D}} \mathbb{E}(w^\tau(t))\eta \, dx \, d\xi + O(\sqrt{\tau}).$$

This shows that $\int_{\mathbb{R}^{2D}} \mathbb{E}(w^\tau(t))\eta \, dx \, d\xi$ is equicontinuous on $[0, \infty)$. Then, by the Ascoli theorem there exist subsequences $\tau_k \to 0$ (again denoted by τ in the sequel for the sake of legibility) such that for any $T > 0$

$$\mathbb{E}(w^\tau(t)) \to w^0(t) \quad \text{in } C^0([0, T]; L^2(\mathbb{R}^{2D}) - weak). \tag{41}$$

We refer to [17,24] for more details.

In order to pass to the limit in (39), there remains to compute the limit of $L_t^\tau(\mathbb{E}(w^\tau(t)))$. Using that θ_t^τ is a skew operator (Proposition 3.0) and that the adjoint of S_t is S_{-t}, the adjoint $(L_t^\tau)^*$ is given by

$$(L_t^\tau)^* = \int_0^1 \mathbb{E}(\tau S_{\sigma\tau}\theta[U^\tau(t - \sigma\tau)]S_{-\sigma\tau}\theta[U^\tau(t)]) \, d\sigma. \tag{42}$$

Therefore, there only remains to obtain the $L^2(\mathbb{R}^{2D})$ strong convergence of $(L_t^\tau)^*(\eta)$ for any test function η. By using (21) a short computation leads to

$$S_{\sigma\tau}\theta[U^\tau(t - \sigma\tau)]S_{-\sigma\tau}\theta[U^\tau(t)](\eta)(x, \xi)$$

$$= -\sum_{\epsilon_1, \epsilon_2 = \pm 1} \frac{\epsilon_1 \epsilon_2}{(2\pi)^{2D}} \int_{\mathbb{R}^{2D}} \left\{ \mathcal{V}_{t-\sigma\tau}^\tau(q) \mathcal{V}_t^\tau(p) \right.$$

$$\left. \times \eta\left(x + \epsilon_2 \tau\sigma \frac{q}{2}, \xi + \epsilon_1 \frac{p}{2} + \epsilon_2 \frac{q}{2}\right) e^{ix\cdot(p+q)} e^{-i\epsilon_2 \tau\sigma q \cdot (\xi - \frac{p}{2})} \right\} dp \, dq.$$

Then, the identity (22) yields

$$(L_t^\tau)^*(\eta) = -\sum_{\epsilon_1, \epsilon_2 = \pm 1} \frac{\epsilon_1 \epsilon_2}{(2\pi)^D} \int_{\mathbb{R}^D \times \mathbb{R}^D} \int_0^1 \hat{R}(\sigma, p, q) \eta\left(x + \epsilon_2 \tau\sigma \frac{q}{2}, \xi \right.$$

$$\left. + \frac{\epsilon_1 p + \epsilon_2 q}{2}\right) e^{ix\cdot(p+q)} e^{-i\epsilon_2 \tau\sigma q \cdot (\xi - \frac{p}{2})} \, d\sigma \, dp \, dq \tag{43}$$

or equivalently

$$(L_t^\tau)^*(\eta) = -\sum_{\epsilon_1, \epsilon_2 = \pm 1} \frac{\epsilon_1 \epsilon_2}{(2\pi)^D} \int_{\mathbb{R}^D \times \mathbb{R}^D} \int_0^1 \hat{R}(\sigma, p, q) \, d\sigma$$

$$\times \eta\left(x, \xi + \frac{\epsilon_1 p + \epsilon_2 q}{2}\right) e^{ix\cdot(p+q)} \, dp \, dq + r_1 + r_2, \tag{44}$$

where

$$
r_1 = - \sum_{\epsilon_1, \epsilon_2 = \pm 1} \frac{\epsilon_1 \epsilon_2}{(2\pi)^D} \int_{\mathbb{R}^D \times \mathbb{R}^D} \int_0^1 \hat{R}(\sigma, p, q) \, \eta \left(x + \epsilon_2 \tau \sigma \frac{q}{2}, \xi + \frac{\epsilon_1 p + \epsilon_2 q}{2} \right)
$$
$$
\times e^{i x \cdot (p+q)} \big[e^{-i \epsilon_2 \tau \sigma q \cdot (\xi - p/2)} - 1 \big] \, d\sigma \, dp \, dq
$$

and

$$
r_2 = - \sum_{\epsilon_1, \epsilon_2 = \pm 1} \frac{\epsilon_1 \epsilon_2}{(2\pi)^D} \int_{\mathbb{R}^D \times \mathbb{R}^D} \int_0^1 \hat{R}(\sigma, p, q) \, e^{i x \cdot (p+q)}
$$
$$
\times \int_0^1 \nabla_x \eta \left(x + \epsilon_2 s \tau \sigma \frac{q}{2}, \xi + \frac{\epsilon_1 p + \epsilon_2 q}{2} \right) \cdot \left(\epsilon_2 \tau \sigma \frac{q}{2} \right) ds \ d\sigma \, dp \, dq.
$$

The remainders r_1, r_2 can be estimated as follows:

$$
|r_1| \leq \sum_{\epsilon_1, \epsilon_2 = \pm 1} \frac{1}{(2\pi)^D} \int_{\mathbb{R}^D \times \mathbb{R}^D} \int_0^1 |\hat{R}(\sigma, p, q)| \left| \eta \left(x + \epsilon_2 \tau \sigma \frac{q}{2}, \xi + \frac{\epsilon_1 p + \epsilon_2 q}{2} \right) \right|
$$
$$
\times \tau |q| \left| \xi + \frac{p}{2} \right| d\sigma \, dp \, dq
$$
$$
\leq \sum_{\epsilon_1, \epsilon_2 = \pm 1} \frac{\tau}{(2\pi)^D} \int_{\mathbb{R}^D \times \mathbb{R}^D} \int_0^1 |\hat{R}(\sigma, p, q)| \left| \eta \left(x + \epsilon_2 \tau \sigma \frac{q}{2}, \xi + \frac{\epsilon_1 p + \epsilon_2 q}{2} \right) \right|
$$
$$
\times \left| \xi + \frac{\epsilon_1 p + \epsilon_2 q}{2} \right| \ |q| \left| \frac{p - \epsilon_1 p - \epsilon_2 q}{2} \right| d\sigma \, dp \, dq.
$$

Now, taking L^2 norms we have

$$
\| r_1 \| \leq C \tau \| |\xi| \eta \| \int_{\mathbb{R}^D \times \mathbb{R}^D} \int_0^1 |\hat{R}(\sigma, p, q)| \, |q| (|p| + |q|) \, d\sigma \, dp \, dq.
$$

Similarly, for r_2 we obtain

$$
\| r_2 \| \leq C \tau \| \nabla_x \eta \| \int_{\mathbb{R}^D \times \mathbb{R}^D} \int_0^1 |\hat{R}(\sigma, p, q)| \, |q| \, d\sigma \, dp \, dq.
$$

Therefore, using assumption (7) and (44) we find

$$
(L_t^\tau)^*(\eta) = \frac{1}{(2\pi)^D} \int_{\mathbb{R}^D \times \mathbb{R}^D} \int_0^1 \hat{R}(\sigma, p, q) \, d\sigma \left(\eta \left(x, \xi + \frac{p - q}{2} \right) \right.
$$
$$
\left. + \eta \left(x, \xi - \frac{p - q}{2} \right) \right) e^{i x \cdot (p+q)} \, dp \, dq
$$

$$- \frac{1}{(2\pi)^D} \int_{\mathbb{R}^D \times \mathbb{R}^D} \int_0^1 \hat{R}(\sigma, p, q) \, d\sigma \left(\eta \left(x, \xi + \frac{p+q}{2} \right) \right.$$

$$\left. + \eta \left(x, \xi - \frac{p+q}{2} \right) \right) e^{ix \cdot (p+q)} \, dp \, dq \; + \; O(\tau).$$

Playing with the symmetries $p \rightleftharpoons q$ and using
$\hat{R}(\sigma, q, p) = \hat{R}(-\sigma, p, q)$ we get

$$(L_t^\tau)^*(\eta) = \frac{1}{2(2\pi)^D} \int_{\mathbb{R}^D \times \mathbb{R}^D} \int_0^1 (\hat{R}(\sigma, p, q) + \hat{R}(-\sigma, p, q)) \, d\sigma \, e^{ix \cdot (p+q)}$$

$$\times \left(\eta \left(x, \xi + \frac{p-q}{2} \right) - \eta \left(x, \xi - \frac{p-q}{2} \right) - \eta \left(x, \xi + \frac{p+q}{2} \right) \right.$$

$$\left. + \eta \left(x, \xi - \frac{p+q}{2} \right) \right) dp \, dq + \; O(\tau)$$

$$= \frac{1}{2(2\pi)^D} \int_{\mathbb{R}^D \times \mathbb{R}^D} \hat{S}(p, q) e^{ix \cdot (p+q)} \left(\eta \left(x, \xi + \frac{p-q}{2} \right) - \eta \left(x, \xi - \frac{p-q}{2} \right) \right.$$

$$\left. - \eta \left(x, \xi + \frac{p+q}{2} \right) + \eta \left(x, \xi - \frac{p+q}{2} \right) \right) dp \, dq \; + \; O(\tau)$$

Making the change of variables $(p, q) \mapsto (-p, -q)$ in the second and fourth
terms and using $\hat{S}(-p, -q) = \overline{\hat{S}(p, q)}$ yields

$$(L_t^\tau)^*(\eta) = \frac{1}{(2\pi)^D} \int_{\mathbb{R}^D \times \mathbb{R}^D} \Re \left(\hat{S}(p, q) e^{ix \cdot (p+q)} \right)$$

$$\times \left(\eta \left(x, \xi + \frac{p-q}{2} \right) - \eta \left(x, \xi + \frac{p+q}{2} \right) \right) dp \, dq \; + \; O(\tau),$$

where we denoted by $\Re$ the real part of the complex quantity.

We can rewrite L_t^τ as follows

$$(L_t^\tau)^*(\eta) = \frac{1}{(2\pi)^D} \int_{\mathbb{R}^D \times \mathbb{R}^D} \Re \left(\hat{S}(p+q, q) e^{ix \cdot (p+2q)} \right) \eta \left(x, \xi + \frac{p}{2} \right) dp \, dq$$

$$- \frac{1}{(2\pi)^D} \int_{\mathbb{R}^D \times \mathbb{R}^D} \Re \left(\hat{S}(p-q, q) e^{ix \cdot p} \right) \eta \left(x, \xi + \frac{p}{2} \right) dp \, dq \; + \; O(\tau)$$

$$= \frac{2^D}{(2\pi)^D} \int_{\mathbb{R}^D} \Re \left(\int_{\mathbb{R}^D} \hat{S}(2(\xi' - \xi) + q, q) e^{ix \cdot (2(\xi' - \xi) + 2q)} \, dq \right) \eta(x, \xi') \, d\xi'$$

$$- \frac{2^D}{(2\pi)^D} \int_{\mathbb{R}^D} \Re \left(\int_{\mathbb{R}^D} \hat{S}(2(\xi' - \xi) - q, q) e^{ix \cdot 2(\xi' - \xi)} \, dq \right) \eta(x, \xi') \, d\xi' + O(\tau),$$

which gives

$$(L_t^\tau)^*(\eta) = \int_{\mathbb{R}^D} (K_2(x, \xi' - \xi) - K_1(x, \xi' - \xi))\, \eta(x, \xi')\, d\xi' \, + \, O(\tau),$$

where

$$K_1(x, p) = \frac{2^D}{(2\pi)^D} \Re \left(\int_{\mathbb{R}^D} \hat{S}(2p - q, q)\, e^{ix\cdot 2p}\, dq \right) \tag{45}$$

$$= \frac{2^D}{(2\pi)^D} \Re \left(\int_{\mathbb{R}^D} \int_{\mathbb{R}^{2D}} S(y, z)\, e^{-i2p\cdot y} e^{iq\cdot y} e^{-iq\cdot z} e^{ix\cdot 2p}\, dy\, dz\, dq \right)$$

$$= \frac{2^D}{(2\pi)^D} \Re \left(\int_{\mathbb{R}^D} \int_{\mathbb{R}^{2D}} S(y, y + z)\, e^{-i2p\cdot y} e^{-iq\cdot z} e^{ix\cdot 2p}\, dy\, dz\, dq \right)$$

$$= 2^D \Re \left(\int_{\mathbb{R}^D} S(y, y)\, e^{-i2p\cdot y} e^{ix\cdot 2p}\, dy \right)$$

$$= \int_{\mathbb{R}^D} S\left(x + \frac{y}{2}, x + \frac{y}{2}\right) \cos(p \cdot y)\, dy, \tag{46}$$

$$K_2(x, p) = \frac{2^D}{(2\pi)^D} \Re \left(\int_{\mathbb{R}^D} \hat{S}(2p + q, q)\, e^{ix\cdot(2p+2q)}\, dq \right) \tag{47}$$

$$= \frac{2^D}{(2\pi)^D} \Re \left(\int_{\mathbb{R}^D} \int_{\mathbb{R}^{2D}} S(y, z)\, e^{-iy\cdot(2p+q)} e^{-iz\cdot q} e^{ix\cdot(2p+2q)}\, dy\, dz\, dq \right)$$

$$= \frac{2^D}{(2\pi)^D} \Re \left(\int_{\mathbb{R}^D} \int_{\mathbb{R}^{2D}} S(y, z - y)\, e^{-i2p\cdot(y-x)} e^{-iz\cdot q}\, dy\, dz e^{i2x\cdot q}\, dq \right)$$

$$= 2^D \Re \left(\int_{\mathbb{R}^D} \int_{\mathbb{R}^{2D}} S(y, 2x - y)\, e^{-i2p\cdot(y-x)}\, dy \right)$$

$$= \Re \left(\int_{\mathbb{R}^D} \int_{\mathbb{R}^{2D}} S(x + \frac{y}{2}, x - \frac{y}{2})\, e^{-ip\cdot y}\, dy \right)$$

$$= \int_{\mathbb{R}^D} \int_{\mathbb{R}^{2D}} S\left(x + \frac{y}{2}, x - \frac{y}{2}\right) \cos(p \cdot y)\, dy$$

There remains to compute K_1 and K_2 in the particular case $R(t, x, y) = Q(t, x - y)$. We start from (45) and use that $\hat{S}(p, q) = (2\pi)^D \int_{-1}^{1} \hat{Q}(\sigma, p)\, d\sigma\, \delta(q),$

$$K_1(x, p) = 2^D \Re \left(\int_{\mathbb{R}^D} \int_{-1}^{1} \hat{Q}(\sigma, 2p - q)\, d\sigma \delta(2p)\, e^{i2p\cdot x}\, dq \right)$$

$$= \Re \int_{\mathbb{R}^D} \int_{-1}^{1} \hat{Q}(\sigma, q)\, d\sigma)\, dq \delta(p) = \Lambda \delta(p),$$

and from (47)

$$K_2(x, p) = 2^D \Re \left(\int_{\mathbb{R}^D} \int_{-1}^{1} \hat{Q}(\sigma, 2p + q) \, d\sigma \, \delta(2p + 2q) \, e^{i2(p+q)\cdot x} \, dq \right)$$

$$= \int_{-1}^{1} \hat{Q}(\sigma, p) \, d\sigma = k(p)$$

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